SIRURI FUNDAMENTALE
(SIRURI COUCHY)

Definitia 1:

Spunem ca un sir a_n este fundamental (sau sir Couchy) daca

$(\forall)\varepsilon > 0, (\exists)N = N(\varepsilon)$ astfel incat $|a_n - a_m| < \varepsilon, (\forall)n, m \geq N(\varepsilon)$.

Definitia 2:

Spunem ca un sir a_n este fundamental (sau sir Couchy) daca

$(\forall)\varepsilon > 0, (\exists)N = N(\varepsilon)$ astfel incat $|a_{n+p} - a_n| < \varepsilon, (\forall)n \geq N(\varepsilon)$ si $p \in N^*$.

Definitia 3:

Spunem ca un sir a_n este fundamental (sau sir Couchy) daca

$(\forall)\varepsilon > 0, (\exists)N = N(\varepsilon)$ astfel incat $|a_n - a_N| < \varepsilon, (\forall)n \geq N(\varepsilon)$.

Observtie!

Cele trei definitii date sunt echivalente:

$$\text{Defintia } 1 \Leftrightarrow \text{Definitia } 2 \Leftrightarrow \text{Definitia} 3$$

Criteriul lui Couchy:

Un sir de numere reale este convergent daca si numai daca este sir Couchy.

$$(a_n) \text{ sir fundamental} \Leftrightarrow (a_n) \text{ este convergent}$$

Problem propuse spre rezolvare:

I. Utilizand criteriul lui Couchy sa se arate ca urmatoarele siruri sunt convergente:

1) $a_n = \dfrac{2n+1}{5n+2}$

Rezolvare:

Demonstram ca

$(\forall)\varepsilon > 0, (\exists)N = N(\varepsilon)$ astfel incat $\left|a_{n+p} - a_n\right| < \varepsilon, (\forall)n \geq N(\varepsilon)$ si $p \in N^*$

$$\left|a_{n+p} - a_n\right| = \frac{2(n+p)+1}{5(n+p)+2} - \frac{2n+1}{5n+2} =$$

$$= \frac{(5n+2)(2n+2p+1) - (2n+1)(5n+5p+2)}{(5n+2)(5n+5p+2)} =$$

$$= \frac{10n^2 + 10np + 9n + 4p + 2 - 10n^2 - 10np - 9n - 5p - 2}{(5n+2)(5n+5p+2)} =$$

$$= \frac{p}{(5n+2)(5n+5p+2)}$$

$$\left|a_{n+p} - a_n\right| < \varepsilon \Rightarrow \frac{p}{(5n+2)(5n+5p+2)} < \varepsilon$$

$$(\forall)\varepsilon > 0, (\exists)N = 1 + \left[\frac{1-10\varepsilon}{25\varepsilon}\right] \text{astfel incat} \left|a_{n+p} - a_n\right| < \varepsilon, (\forall)n \geq N(\varepsilon) \text{si } p \in N$$

2) $$b_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \Lambda + \frac{1}{2^n}$$

Demonstram ca

$$(\forall)\varepsilon > 0, (\exists)N = N(\varepsilon) \text{astfel incat} \left|b_{n+p} - b_n\right| < \varepsilon, (\forall)n \geq N(\varepsilon) \text{si } p \in N$$

$$\left|b_{n+p} - b_n\right| = 1 + \frac{1}{2} + \frac{1}{2^2} + \Lambda + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \Lambda + \frac{1}{2^{n+p}} - 1 - \frac{1}{2} - \frac{1}{2^2} - \Lambda - \frac{1}{2^n} =$$

$$= \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \Lambda + \frac{1}{2^{n+p}} = \frac{1}{2^{n+1}}\left(1 + \frac{1}{2} + \Lambda + \frac{1}{2^{p-1}}\right) < \frac{1}{2^n}\left(\frac{1}{2} + \frac{1}{2^2} + \Lambda + \frac{1}{2^p}\right) =$$

$$= \frac{1}{2^n} \cdot \frac{\frac{1}{2}\left(1 - \frac{1}{2^p}\right)}{1 - \frac{1}{2}} = \frac{1}{2^n}\left(1 - \frac{1}{2^p}\right) < \frac{1}{2^n}$$

$$\Rightarrow \left.\begin{array}{l} \left|b_{n+p} - b_n\right| < \frac{1}{2^n} \\ \left|b_{n+p} - b_n\right| < \varepsilon \end{array}\right\} \Rightarrow \frac{1}{2^n} < \varepsilon \Rightarrow n > \log_2 \frac{1}{\varepsilon}$$

Deci $N(\varepsilon) = 1 + \left[\log_2 \frac{1}{\varepsilon}\right]$

$$(\forall)\varepsilon > 0, (\exists)N(\varepsilon) = 1 + \left[\log_2 \frac{1}{\varepsilon}\right] \text{astfel incat} \left|b_{n+p} - b_n\right| < \varepsilon, (\forall)n \geq N(\varepsilon) \text{si } p \in N$$

p este un numar arbirar. Cind $p \to \infty$ obtinem:

$$\lim_{p\to\infty} \frac{p}{(5n+2)(5n+5p+2)} = \lim_{p\to\infty} \frac{p}{p(5n+2)\left(\frac{5n}{p} + 5 + \frac{2}{p}\right)} = \frac{1}{5(5n+2)}$$

$$\Rightarrow \frac{1}{5(5n+2)} < \varepsilon \Rightarrow n \geq \frac{1-10\varepsilon}{25\varepsilon}$$, deci putem lua $N(\varepsilon) = 1 + \left[\frac{1-10\varepsilon}{25\varepsilon}\right]$

3) $c_n = \sum_{k=1}^{n} \frac{\sin k^2}{2^k}$

$$\left|c_{n+p} - c_n\right| = \left|\sum_{k=1}^{n+p} \frac{\sin k^2}{2^k} - \sum_{k=1}^{n} \frac{\sin k^2}{2^k}\right| = \left|\sum_{k=n+1}^{n+p} \frac{\sin k^2}{2^k}\right| \le \sum_{k=n+1}^{n+p} \left|\frac{\sin k^2}{2^k}\right| \le$$

$$\le \sum_{k=n+1}^{n+p} \frac{1}{2^k} = \frac{1}{2^{n+1}}\left(1 + \frac{1}{2} + \Lambda + \frac{1}{2^{p-1}}\right) < \frac{1}{2^n}$$

dupa cum am aratat la exercitiul anterior $\Rightarrow$

$\Rightarrow$ se obtine $N(\varepsilon) = 1 + \left[\log_2 \frac{1}{\varepsilon}\right]$

$$(\forall)\varepsilon > 0, (\exists)N(\varepsilon) = 1 + \left[\log_2 \frac{1}{\varepsilon}\right] \text{astfel incat} \left|b_{n+p} - b_n\right| < \varepsilon, (\forall)n \ge N(\varepsilon) \text{si } p \in N$$

4) $d_n = \sum_{k=1}^{n} \frac{\cos k!}{k(k+1)}$

$$\left|d_{n+p} - d_n\right| = \left|\sum_{k=1}^{n+p} \frac{\cos k!}{k(k+1)} - \sum_{k=1}^{n} \frac{\cos k!}{k(k+1)}\right| = \left|\sum_{k=n+1}^{n+p} \frac{\cos k!}{k(k+1)}\right| \le$$

$$\le \sum_{k=n+1}^{n+p} \left|\frac{\cos k!}{k(k+1)}\right| < \sum_{k=n+1}^{n+p} \frac{1}{k(k-1)} = \sum_{k=n+1}^{n+p} \frac{1}{k} - \frac{1}{k-1} =$$

$$= \frac{1}{n+1} - \frac{1}{n+p+1} < \frac{1}{n+1}$$

$$\Rightarrow \left.\begin{array}{r} \left|d_{n+p} - d_n\right| < \dfrac{1}{n+1} \\ \left|d_{n+p} - d_n\right| < \varepsilon \end{array}\right\} \Rightarrow \frac{1}{n+1} \le \varepsilon \Rightarrow N(\varepsilon) = 1 + \left[\frac{1-\varepsilon}{\varepsilon}\right]$$

II. Aratati ca urmatorul sir de numere reale nu este fundamental:

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \Lambda + \frac{1}{n}$$

Aratam ca :

$$(\exists)\varepsilon > 0, (\forall)N = N(\varepsilon) \text{ astfel incat } |x_{n+p} - x_n| \geq \varepsilon, (\forall)n \geq N(\varepsilon) \text{ si } p \in N^*$$

$$|x_{2n} - x_n| = \left|\sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k}\right| = \left|\sum_{k=n+1}^{2n} \frac{1}{k}\right| \leq \sum_{k=n+1}^{2n} \left|\frac{1}{k}\right| = \sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \Lambda + \frac{1}{2n} >$$

$$> \frac{1}{2n} + \frac{1}{2n} + \Lambda + \frac{1}{2n} = n\frac{1}{2n} = \frac{1}{2}$$

$$|x_{2n} - x_n| > \frac{1}{2} \Rightarrow$$

$$\Rightarrow (\exists)\varepsilon = \frac{1}{3}, (\forall)N = N(\varepsilon) \text{ astfel incat } |x_{n+p} - x_n| \geq \varepsilon, (\forall)n \geq N(\varepsilon)$$

$\Rightarrow$ <u>sirul x_n nu este fundamental</u>