RELATII DE RECULENTA LA SIRURI

O relatie ce se stabileste intre doi sau mai multi termeni consecutivi ai unui sir, se numeste **relatie de reculenta.**

Relatia de reculenta poate fi data sub mai multe forme:

a) sub forma explicita:

$$a_{n+1} = f(a_n) \text{ - relatie de reculenta de ordinul } 1.$$

$$a_{n+2} = f(a_{n+1}, a_n) \text{ - relatie de reculenta de ordinul } 2.$$

$$a_{n+k} = f(a_{k+1}, \ldots a_{n+k-1}) \text{ - relatie de reculenta de ordinul k.}$$

b) sub forma implicita:

$$f(a_{n+1}, a_n) = 0 \text{ - relatie de reculenta de ordinul } 1.$$

$$f(a_{n+2}, a_{n+1}, a_n) = 0 \text{ - relatie de reculenta de ordinul } 2$$

$$f(a_{n+k}, \ldots a_n) = 0 \text{ - relatie de reculenta de ordinul k.}$$

ex: -explicita:

$$a_n = \frac{2}{7} a_{n-1}, \; n \geq 1 \; a_0 = 3$$

$$a_{n+2} = \frac{a_{n+1} + a_n}{4}, \text{ se dau } a_1 = 2, a_2 = 3$$

-implicita:

$$7a_n - 2a_{n-1} = 0$$

$$2a_{n+2} - a_{n+1} - a_n = 0$$

$$a_n^2 - 2a_{n+1} - 3 = 0$$

Relatiide reculenta liniare de ordinul I

$a_{n+1} = a_n + r$, se da a_1

$$a_2 = a_1 + r$$
$$a_3 = a_2 + r$$
$$a_4 = a_3 + r$$
$$a_n = a_{n-1} + r$$
$$\overline{a_n = a_1 + (n-1)r}$$

$a_{n+1} = a_n \cdot q$, se da a_1, q, $q \neq 0$

$$a_2 = a_1 \cdot q$$
$$a_3 = a_2 \cdot q$$
$$a_n = a_{n-1} \cdot q$$
$$\overline{a_n = a_1 \cdot q^{n-1}}$$

$a_{n+1} = \alpha \cdot a_n + \beta$ se da $a_1, \alpha, \beta \in$ R

$$a_2 = \alpha \cdot a_1 + \beta \quad \Big| \cdot \alpha^{n-2} \qquad \alpha^{n-2} a_2 = \alpha^{n-1} a_1 + \beta \alpha^{n-2}$$
$$a_3 = \alpha \cdot a_2 + \beta \quad \Big| \cdot \alpha^{n-3} \qquad \alpha^{n-3} a_3 = \alpha^{n-2} a_2 + \beta \alpha^{n-3}$$
$$a_4 = \alpha \cdot a_3 + \beta \quad \Big| \cdot \alpha^{n-4} \qquad \alpha^{n-4} a_4 = \alpha^{n-3} a_3 + \beta \alpha^{n-4}$$

$$a_{n-2} = \alpha \cdot a_{n-3} + \beta \quad \Big| \cdot \alpha^{2} \qquad \alpha^{2} a_{n-2} = \alpha^{3} a_{n-3} + \beta \alpha^{2}$$
$$a_{n-1} = \alpha \cdot a_{n-2} + \beta \quad \Big| \cdot \alpha \qquad \alpha a_{n-1} = \alpha^{2} a_{n-2} + \beta \alpha$$
$$a_n = \alpha \cdot a_{n-1} + \beta \quad \Big| \qquad a_n = \alpha a_{n-1} + \beta$$

$$a_n = \alpha^{n-1} a_1 + \beta \left(\alpha^{n-2} + \alpha^{n-1} + \ldots + 1\right)$$

I. $\alpha = 1$

$$a_n = a_1 + (n-1)\beta$$

II. $\alpha \neq 1$

$$a_n = \alpha^{n-1} a_1 + \beta \frac{1-\alpha^{n-1}}{1-\alpha}$$

III. $\beta = 0$

$$a_n = \alpha^{n-1} a_1$$

Exemplu:

$$a_n = 2a_{n-1} + 3 \quad ; \quad a_1 = 2 \quad n \geq 2$$

$$\begin{array}{l|l} a_2 = 2a_1 + 3 & 2^{n-2} \\ a_3 = 2a_2 + 3 & 2^{n-3} \\ a_4 = 2a_3 + 3 & 2^{n-4} \\ & \\ a_{n-2} = 2a_{n-3} + 3 & 2^2 \\ a_{n-1} = 2a_{n-2} + 3 & 2 \\ a_n = 2a_{n-1} + 3 & \end{array}$$

$$\begin{array}{l|} 2^{n-2}a_2 = 2^{n-1}a_1 + 3 \cdot 2^{n-2} \\ 2^{n-3}a_3 = 2^{n-2}a_2 + 3 \cdot 2^{n-3} \\ 2^{n-4}a_4 = 2^{n-3}a_3 + 3 \cdot 2^{n-4} \\ \\ 2^2 a_{n-2} = 2^3 a_{n-3} + 3 \cdot 2^2 \\ 2a_{n-1} = 2^2 a_{n-2} + 3 \cdot 2 \\ a_n = 2a_{n-1} + 3 \\ \hline \end{array}$$

$$a_n = 2^{n-1} \cdot 2 + 3\left(1 + 2 + 2^2 + \ldots + 2^{n-2}\right)$$

$$a_n = 2^n + 3\,\frac{2^{n-1} - 1}{2 - 1}$$

$$\boxed{a_n = 2^{n-1} \cdot 5 + 3}$$

Relatii de reculenta de ordinul II, liniare si omogene:

$$\alpha \cdot a_{n+2} + \beta \cdot a_{n+1} + \gamma \cdot a_n = 0$$ se dau doi termeni : a_1, a_2 si $\alpha, \beta, \gamma \in \mathbb{R}$

Notam:

$a_n = c \cdot k^n$

$a_{n+1} = c \cdot k^{n+1}$

$a_{n+2} = c \cdot k^{n+2}$

$\alpha \cdot c \cdot k^{n+2} + \beta \cdot c \cdot k^{n+1} + \gamma \cdot c \cdot k^n = 0$

$c \cdot k^n\left(\alpha \cdot k^2 + \beta \cdot k + \gamma\right) = 0 \qquad c \cdot k^n \neq 0$

$\Rightarrow \alpha \cdot k^2 + \beta \cdot k + \gamma = 0$

k radacina a ecuatiei $\alpha \cdot k^2 + \beta \cdot k + \gamma = 0$ – ecuatia caracteristica atasata relatiei de reculenta.

Caz I : $\Delta > 0$

$k_{1,2}$ = doua radacini reale si distincte;

$a_n = c_1 \cdot k_1^n + c_2 \cdot k_2^n$

$\begin{cases} a_1 = c_1 \cdot k_1 + c_2 \cdot k_2 \\ a_2 = c_1 \cdot k_1^2 + c_2 \cdot k_2^2 \end{cases} \Rightarrow c_1, c_2$ care se inlocuiesc in $a_n = c_1 \cdot k_1^n + c_2 \cdot k_2^n$

Caz II : $\Delta = 0$

$k_1 = k_2 = k$ = radacini reale si egale

$a_n = k^n\left(c_1 \cdot n + c_2\right)$

$\begin{cases} a_1 = k\left(c_1 + c_2\right) \\ a_2 = k^2\left(c_1 \cdot n + c_2\right) \end{cases} \Rightarrow c_1, c_2$

Caz III : $\Delta < 0$

$k_1, k_2 \in \mathbb{C} \setminus \mathbb{R} \qquad k_{1,2} = \dfrac{-\beta \pm i\sqrt{-\Delta}}{2\alpha}$

$a_n = r^n\left(c_1 \cdot \cos nt + c_2 \sin nt\right) \quad r = |k_1| = |k_2| \quad \operatorname{tg} t = \dfrac{b}{a}$

$\begin{cases} a_1 = r\left(c_1 \cos t + c_2 \sin t\right) \\ a_2 = r^2\left(c_1 \cos 2t + c_2 \sin 2t\right) \end{cases} \Rightarrow c_1, c_2$

Exemplu: Sa se afle termenul general al sirului, daca se da urmatoarea relatie de reculenta:

$$a_{n+2} - 4a_{n+1} + 4a_n = 0 \qquad , \qquad a_1 = 2 \, , a_2 = 3$$

$$a_n = c \cdot k^n$$

$$a_{n+1} = c \cdot k^{n+1}$$

$$a_{n+2} = c \cdot k^{n+2}$$

$$c \cdot k^{n+2} - 4c \cdot k^{n+1} + 4c \cdot k^n = 0$$

$$c \cdot k^n \left(k^2 - 4k + 4\right) = 0$$

$$k^2 - 4k + 4 = 0 \Rightarrow \Delta = 0$$

$$k_{1,2} = \frac{4}{2} = 2 \Rightarrow \underline{k_{1,2} = 2 = k}$$

$$a_n = k^n \left(c_1 \cdot n + c_2\right)$$

$$\begin{cases} a_1 = k\left(c_1 + c_2\right) \\ a_2 = k^2 (2c_1 + c_2) \end{cases} \Leftrightarrow \begin{cases} 2 = 2\left(c_1 + c_2\right) \\ 3 = 4\left(2c_1 + c_2\right) \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = 1 \\ 8c_1 + 4c_2 = 3 \end{cases}$$

$$\Rightarrow c_1 = -\frac{1}{4} \, , \, c_2 = \frac{5}{4}$$

$$\boxed{a_n = 2^n \left(-\frac{1}{4} n + \frac{5}{4}\right)}$$

Relatii de reculenta de ordinul II, liniare si neomogene:

$$\alpha \cdot a_{n+2} + \beta \cdot a_{n+1} + \gamma \cdot a_n = f(n) \quad \text{se dau } \; a_0 = 0 \; \text{ si } \; a = 1$$

Se transforma relatia de mai sus intr-o relatie de reculenta liniara si omogena:

$$a_{n+2} - 7a_{n+1} + 12a_n = 6n + 1 \qquad \begin{cases} a_0 = 0 \\ a_1 = 1 \end{cases}$$

$$Notam:$$

$$a_n = b_n + \alpha \cdot n + \beta$$

$$a_{n+1} = b_{n+1} + \alpha \cdot (n+1) + \beta$$

$$a_{n+2} = b_{n+2} + \alpha \cdot (n+2) + \beta$$

$$b_{n+2}+\alpha(n+2)+\beta-7[b_{n+1}+\alpha\cdot(n+1)+\beta]+12(b_n+\alpha\cdot n+\beta)=6n+1$$
$$b_{n+2}-7b_{n+1}+12b_n+6\alpha\cdot n-5\alpha+6\beta=6n+1$$

$$\begin{cases}6\alpha\cdot n=6n\\-5\alpha+6\beta=1\end{cases}\Rightarrow\begin{cases}\alpha=1\\\beta=1\end{cases}$$

$$a_n=b_n+\alpha\cdot n+\beta$$

$$\Rightarrow\begin{matrix}a_0=b_0+\beta\\0=b_0+1\Rightarrow \underline{b_0=-1}\end{matrix}\quad\Bigg|\quad\begin{matrix}a_1=b_1+\alpha+\beta\\1=b_1+2\Rightarrow b_1=-1\end{matrix}$$

$$b_{n+2}-7b_{n+1}+12b_n=0$$
$$b_n=c\cdot k^n$$
$$b_{n+1}=c\cdot k^{n+1}$$
$$b_{n+2}=c\cdot k^{n+2}$$
$$c\cdot k^n(k^2-7k+12)=0$$
$$k_{1,2}=\left\langle\begin{matrix}4\\3\end{matrix}\right.$$
$$b_n=c_1k_1^n+c_2k_2^n$$
$$\begin{cases}b_0=c_1+c_2\\b_1=c_1k_1+c_2k_2\end{cases}\Leftrightarrow\begin{cases}-1=c_1+c_2\\-1=4c_1+3c_2\end{cases}\Leftrightarrow\begin{cases}c_1=-3\\c_2=2\end{cases}$$
$$\Rightarrow b_n=2\cdot4^n-3\cdot3^n$$
$$b_n=2^{n+1}-3^{n+1}$$
$$a_n=b_n+\alpha\cdot n+\beta\Rightarrow\underline{a_n=2^{n+1}-3^{n+1}+n+1}$$