• Sirul lui Euler

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

Se demonstreaza convergenta :

$$a_{n+1} = (1+\frac{1}{n+1})^{n+1} = C_{n+1}^0 + ... + C_{n+1}^{n-1}(\frac{1}{n+1})^{n-1} + C_{n+1}^{n}(\frac{1}{n+1})^{n} + C_{n+1}^{n+1}(\frac{1}{n+1})^{n+1}$$

$$a_n = (1+\frac{1}{n})^n = C_n^0 + C_n^1\frac{1}{n} + ... + C_n^{n-2}(\frac{1}{n})^{n-2} + C_n^{n-1}(\frac{1}{n})^{n-1} + C_n^{n}(\frac{1}{n})^{n}$$

#Monotonia

$$a_{n+1} - a_n = ... + C_{n+1}^{n-1}(\frac{1}{n+1})^{n-1} + C_{n+1}^{n}(\frac{1}{n+1})^{n} + C_{n+1}^{n+1}(\frac{1}{n+1})^{n+1}...$$

$$- C_n^{n-2}(\frac{1}{n})^{n-2} - C_n^{n-1}(\frac{1}{n})^{n-1} - C_n^{n}(\frac{1}{n})^{n}$$

$$a_{n+1} - a_n = ...\frac{(n+1)!}{(n-1)!\cdot 2!}\cdot\frac{1}{(n+1)^{n-1}} + \frac{(n+1)!}{n!}\cdot\frac{1}{(n+1)^{n}} + \frac{1}{(n+1)^{n+1}}...$$

$$-\frac{n!}{(n-2)!\cdot 2!}\cdot\frac{1}{n^{n-2}} - \frac{n!}{(n-1)!}\cdot\frac{1}{n^{n-1}} - \frac{1}{n^n}$$

$$a_{n+1} - a_n = ...\frac{n(n+1)}{2!}\cdot\frac{1}{(\mathrm{n}+1)^{\mathrm{n}-1}} + \frac{n+1}{(n+1)^n}\cdot\frac{1}{(\mathrm{n}+1)^{\mathrm{n}+1}}... - \frac{n(n+1)}{2!} - \frac{n}{n^{n-1}} - \frac{1}{n^n} > 0$$

$$\Rightarrow a_{n+1} - a_n > 0 \Rightarrow a_{n+1} > a_n$$

$$a_1 < a_2 < ... < a_n$$

#Marginirea

$$a_n = 1+1+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + ... + \frac{1}{n^n} > 2$$

$$a_n = 1+1+\frac{1}{2!}\cdot\frac{n(n-1)}{n\cdot n} + ... + \frac{1}{k!}\cdot\frac{n(n-1)}{n\cdot n} + ... + \frac{(n-k+1)}{n} + ... + \frac{1}{n^n}$$

$$a_n = 1+1+\frac{1}{2!}\cdot(1-\frac{1}{n}) + ... + \frac{1}{k!}\cdot(1-\frac{1}{n})\cdot(1-\frac{2}{n})\cdot ...\cdot(1-\frac{k-1}{n}) + ... + \frac{1}{n^n}$$

$$a_n < 1+1+\frac{1}{2} + \frac{1}{2\cdot 2} + ... + \frac{1}{2^{n-1}} = 1+2(1-\frac{1}{2^n}) = 3 - \frac{1}{2^{n-1}} < 3$$

$$2 < a_n < 3$$

$$\boxed{\Rightarrow \lim_{n\to\infty}(1+\frac{1}{n})^n = e}$$

$$\bullet \lim_{n\to\infty}(1+\frac{1}{n})^{n+1} = \lim_{n\to\infty}(1+\frac{1}{n})^{n} \cdot \lim_{n\to\infty}(1+\frac{1}{n}) = e\cdot 1 = e$$

$$\boxed{\lim_{n\to\infty}(1+\frac{1}{n})^{n+1} = e}$$

$$\bullet (1+\frac{1}{n})^{n} \xrightarrow{n\to\infty} e \xleftarrow{n\to\infty} (1+\frac{1}{n})^{n+1}$$

$$(1+\frac{1}{n})^{n} < e < (1+\frac{1}{n})^{n+1} \quad \Big| \ln$$

$$\ln(1+\frac{1}{n})^{n} < \ln e < \ln(1+\frac{1}{n})^{n+1} \qquad (\text{Dar } \ln e = 1)$$

$$\text{n} \cdot \ln(\frac{n+1}{n}) < 1 < (n+1)\cdot \ln(\frac{n+1}{n})$$

$$n[\ln(n+1) - \ln n] < 1 < (n+1)[\ln(n+1) - \ln n]$$

$$\begin{cases} \ln(n+1) - \ln n < \dfrac{1}{n} \\ \ln(n+1) - \ln n > \dfrac{1}{n+1} \end{cases}$$

$$\Rightarrow \boxed{\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}}$$

$$\bullet \text{ avem } \lim_{n\to\infty}(1+\frac{1}{n})^{n} = e$$

fie $(x_n)_{n\geq 0}$ sir de nr R, a.î $\text{x}_\text{n} \xrightarrow{n\to\infty} +\infty$

$$\Rightarrow \text{avem } \lim_{n\to\infty}(1+\frac{1}{x_n})^{x_n} = e$$

$\bullet$ fie $y_n \xrightarrow{n\to\infty} -\infty$

notez $x_n = -y_n \Rightarrow -y_n \xrightarrow{n\to\infty} +\infty$

$x_n \xrightarrow{n\to\infty} +\infty$

$$\lim_{n\to\infty}(1+\frac{1}{-x_n})^{-x_n} = \lim_{n\to\infty}(\frac{-x_n+1}{-x_n})^{-x_n} = \lim_{n\to\infty}(\frac{x_n}{x_n-1})^{x_n} =$$

$$=\lim_{n\to\infty}(1+\frac{1}{x_n-1})^{x_n}=\lim_{n\to\infty}[(1+\frac{1}{x_n-1})^{x_n-1}(1+\frac{1}{x_n-1})]=$$

$$=\lim_{n\to\infty}(1+\frac{1}{x_n-1})^{x_n-1}\cdot\lim_{n\to\infty}(1+\frac{1}{x_n-1})=e\cdot 1=e$$

$$y_n\xrightarrow{n\to\infty}-\infty$$

$$\boxed{\lim_{n\to\infty}(1+\frac{1}{y_n})^{y_n}=e}$$

• fie $x_n\xrightarrow{n\to\infty}0$ $\quad x_n>0$

$$\lim_{n\to\infty}(1+x_n)^{\frac{1}{x_n}}=e \qquad notez: x_n=\frac{1}{y_n}$$

$$x_n\xrightarrow{n\to\infty}0$$

$$\Rightarrow \frac{1}{y_n}\xrightarrow{n\to\infty}\infty$$

$$\lim_{n\to\infty}(1+x_n)^{\frac{1}{x_n}}=\lim_{n\to\infty}(1+\frac{1}{y_n})^{y_n}=e$$

$$x_n\xrightarrow{n\to\infty}0 \qquad x_n>0$$

$$\lim_{n\to\infty}(1+x_n)^{\frac{1}{x_n}}=e$$

• fie $x_n\to 0$ $\quad x_n<0$

notez $x_n=-\frac{1}{y_n}$ $\quad x_n\xrightarrow{n\to\infty}0$ $\quad \Rightarrow -\frac{1}{y_n}\xrightarrow{n\to\infty}-\infty$

$$\lim_{n\to\infty}(1+x_n)^{\frac{1}{x_n}}=\lim_{n\to\infty}(1+\frac{1}{-y_n})^{-y_n}=\lim_{n\to\infty}(\frac{1-y_n}{-y_n})^{-y_n}=$$

$$\lim_{n\to\infty}(\frac{y_n}{y_n-1})^{y_n}=\lim_{n\to\infty}(1+\frac{1}{y_n-1})^{y_n}=e$$

$$x_n\xrightarrow{n\to\infty}0 \qquad x_n<0$$

$$\lim_{n\to\infty}(1+x_n)^{\frac{1}{x_n}}=e$$

• fie $(x_n)_{n\geq 0}$ sir de nr R_+ si $x_n \xrightarrow{n\to\infty} +\infty$

$$\frac{e^{x_n}}{x_n} \xrightarrow{n\to\infty} +\infty$$

$e \cong 2{,}71$

pt $m \in N$ $\quad e^m \cong 2{,}71^m = (1+1{,}71)^m \geq 1 + m\cdot 1{,}71 > 1 + m > m$

pt $(x_n)_{n\geq 0}$ $(\exists)$ $m \in N$ a.î $m > x_n$

$e^m > e^{x_n} > m > x_n \Rightarrow e^{x_n} > x_n$

$$\frac{e^{x_n}}{x_n} = \left(\sqrt{\frac{e^{x_n}}{x_n}}\right)^2 = \frac{\sqrt{(e^{x_n})^2}}{\sqrt{(x_n)^2}} = \left(\frac{e^{\frac{x_n}{2}}}{\sqrt{x_n}}\right)^2 \geq \frac{\left(\frac{x_n}{2}\right)^2}{\sqrt{x_n}^2} = \frac{x_n^2}{4}\cdot\frac{1}{x_n} = \frac{x_n}{4}$$

$$\frac{e^{x_n}}{x_n} \geq \frac{x_n}{4} > 0$$

$$\lim_{n\to\infty}\frac{e^{x_n}}{x_n} > \lim_{n\to\infty}\frac{x_n}{4} = +\infty$$

$$\Rightarrow \lim_{n\to\infty}\frac{e^{x_n}}{x_n} = +\infty$$

• fie $(x_n)_{n\geq 0}$ sir de nr R_+ si $x_n \xrightarrow{n\to\infty} +\infty$

$$\lim_{n\to\infty}\frac{\ln x_n}{x_n} = 0$$

$e^{x_n} > x_n \ \big|\ln$

$\ln e^{x_n} > \ln x_n$

$x_n \ln e > \ln x_n$

$x_n > \ln x_n$

$$\frac{\ln x_n}{x_n} = \frac{\ln\sqrt{x_n^2}}{x_n} = 2\frac{\ln\sqrt{x_n}}{x_n} \leq 2\frac{\sqrt{x_n}}{x_n}$$

$$0 \le \frac{\ln x_n}{x_n} \le 2\frac{1}{\sqrt{x_n}} \quad \Big| \lim_{n\to\infty}$$

$$0 < \lim_{n\to\infty}\frac{\ln x_n}{x_n} \le \lim_{n\to\infty}\frac{2}{\sqrt{x_n}}$$

$$0 \le \lim_{n\to\infty}\frac{\ln x_n}{x_n} \le 0$$

$$\Rightarrow \lim_{n\to\infty}\frac{\ln x_n}{x_n} = 0$$

• Sa se calculeze:

$\lim_{n\to\infty}(\sqrt[n]{n!} - \sqrt[n-1]{(n-1)!})$ (Problema lui Traian Lalescu)

Vom folosii Teorema lui D'Alambert:

Daca $(u_n)_{n\in N^*}$ este un sir de numere reale strict pozitive si daca

$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = u > 0$ atunci $\sqrt[n]{u_n} = u$

Fie $u_n = \frac{(n+1)^n}{n!}$ atunci:

$$\frac{u_{n+1}}{u_n} = \frac{(n+2)^{n+1}}{(n+1)!}\cdot\frac{n!}{(n+1)^n} = \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\to \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1} = e$$

Deci $\lim_{n\to\infty}\sqrt[n]{\frac{(n+1)^n}{n!}} = e \ \to \lim_{n\to\infty}\frac{n+1}{\sqrt[n]{n!}} = e$

Fie $u_n = \frac{n^n}{n!}$ atunci :

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = (\frac{n+1}{n})^n = (1+\frac{1}{n})^n$$

$$\rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

Deci $\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} = e$

$$\rightarrow \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$$

Avem :

$$\lim_{n\to\infty}(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}})^n = \lim_{n\to\infty}(\sqrt[n(n+1)]{\frac{[(n+1)!]^n}{(n!)^{n+1}}})^n =$$

$$= \lim_{n\to\infty}(\sqrt[n(n+1)]{\frac{(n+1)^n}{n!}})^n = \lim_{n\to\infty}(\sqrt[n+1]{\frac{n+1}{\sqrt[n]{n!}}})^n =$$

$$= \lim_{n\to\infty}(\frac{n+1}{\sqrt[n]{n!}})^{\frac{n}{n+1}} = \lim_{n\to\infty}(\frac{n+1}{\sqrt[n]{n!}})^{\lim_{n\to\infty}(\frac{n}{n+1})} \quad (\text{Dar} \lim_{n\to\infty}\frac{n}{n+1} = 1) \rightarrow$$

$$\rightarrow \lim_{n\to\infty}(\frac{n+1}{\sqrt[n]{n!}})^{\lim_{n\to\infty}(\frac{n}{n+1})} = \lim_{n\to\infty}\frac{n+1}{\sqrt[n]{n!}} = e$$

Prin urmare :

$$\lim_{n\to\infty}(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}})^n = e$$

$$e = \lim_{n\to\infty}(1+\frac{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}}{\sqrt[n]{n!}})^n =$$

$$= \lim_{n\to\infty}((1+\frac{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}}{\sqrt[n]{n!}})^{\frac{n\sqrt[n]{n!}}{n+1\sqrt{(n+1)!} - \sqrt[n]{n!}}})^{\frac{n+1\sqrt{(n+1)!} - \sqrt[n]{n!}}{\sqrt[n]{n!}} \cdot n}$$

$$\rightarrow e = e^{\lim_{n\to\infty} n+1\sqrt{(n+1)!} - \sqrt[n]{n!} \cdot \frac{\sqrt[n]{n!}}{n}} \rightarrow$$

$$\lim_{n\to\infty}(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!})\cdot\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=1\rightarrow$$

$$\lim_{n\to\infty}(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!})=\frac{1}{\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}}\rightarrow$$

$$\rightarrow\lim_{n\to\infty}(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!})=\frac{1}{e}$$