Metoda de integrare prin parti

Teorema: Daca *f, g*: J $\rightarrow$ **R** sunt functii derivabile cu derivatele continue, atunci functiile *fg ,f'g* si *fg'* admit primitive si multimile lor de primitive sunt legate prin relatia:

$$\int f(x)g'(x)dx = fg - \int f'(x)g(x)dx$$

Demonstratie

$$f, g \text{ derivabile} \rightarrow f, g \text{ continue}$$

$$f', g' \text{ continue } \rightarrow f'g \text{ si } fg' \text{ continue} \rightarrow$$

$$(f'g) + (fg') = (fg)' \rightarrow (fg)' \text{ conntinua}$$

$$\rightarrow f'g, fg', (fg)' \text{ a.p}$$

$$\rightarrow \int (fg)'(x)dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$\text{dar } \int (fg)'(x)dx = (fg)(x) + C \rightarrow$$

$$fg + C = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$\rightarrow \int f(x)g'(x)dx = fg - \int f'(x)g(x)dx + C$$

Exercitii

Sa se calculeze:

$$1)\ I = \int \frac{1 - 2e^x \cos x}{(e^x - \sin x)(e^x - \cos x)} dx$$

$$\begin{aligned}
[(e^x - \sin x)(e^x - \cos x)]' &= (e^x - \sin x)'(e^x - \cos x) + (e^x - \sin x)(e^x - \cos x)' \\
&= (e^x - \cos x)(e^x - \cos x) + (e^x - \sin x)(e^x + \sin x) \\
&= (e^x - \cos x)^2 + (e^{2x} - \sin^2 x) \\
&= e^{2x} - 2e^x \cos x + \cos^2 x + e^{2x} - \sin^2 x \\
&= 2e^{2x} - 2e^x \cos x + 1 - 2\sin^2 x
\end{aligned}$$

$$[(e^x - \cos x)(e^x - \sin x)]' = 1 - 2e^x \cos x + 2(e^{2x} - \sin^2 x)$$

$$[(e^x - \cos x)(e^x - \sin x)]' = 1 - 2e^x \cos x + 2(e^x - \sin x)(e^x + \sin x)$$

$$\Rightarrow 1 - 2e^x \cos x = [(e^x - \cos x)(e^x - \sin x)]' - 2(e^x - \sin x)(e^x + \sin x)$$

$$\int \frac{1 - 2e^x \cos x}{(e^x - \sin x)(e^x - \cos x)} dx =$$

$$I = \int \frac{[(e^x - \cos x)(e^x - \sin x)]' - 2(e^x - \sin x)(e^x + \sin x)}{(e^x - \sin x)(e^x - \cos x)} dx$$

$$I = \int \frac{[(e^x - \sin x)(e^x - \cos x)]'}{(e^x - \sin x)(e^x - \cos x)} dx - 2\int \frac{(e^x - \sin x)(e^x + \sin x)}{(e^x - \sin x)(e^x - \cos x)} dx$$

$$I = \ln\left|(e^x - \sin x)(e^x - \cos x)\right| - 2\int \frac{(e^x + \sin x)}{(e^x - \cos x)} dx$$

$$I = \ln\left|(e^x - \sin x)(e^x - \cos x)\right| - 2\int \frac{(e^x - \cos x)'}{(e^x - \cos x)} dx$$

$$I = \ln\left|(e^x - \sin x)(e^x - \cos x)\right| - 2\ln\left|e^x - \cos x\right| + C$$

2) $I = \displaystyle\int \frac{x e^{\arcsin x}}{\sqrt{1 - x^2}} dx$

$$f(x) = e^{\arcsin x} \qquad g'(x) = \frac{x}{\sqrt{1 - x^2}}$$

$$f'(x) = \frac{e^{\arcsin x}}{\sqrt{1 - x^2}} \qquad g(x) = -\sqrt{1 - x^2}$$

$$\Rightarrow I = -e^{\arcsin x}\sqrt{1 - x^2} + \int e^{\arcsin x} dx \ (I_1)$$

$$I_1 = \int e^{\arcsin x} dx$$

$$I_1 = \int e^{\arcsin x} dx$$

$$f(x) = e^{\arcsin x} \quad g'(x) = 1$$

$$f'(x) = \frac{e^{\arcsin x}}{\sqrt{1-x^2}} \quad g(x) = x$$

$$I_1 = xe^{\arcsin x} - \int \frac{xe^{\arcsin x}}{\sqrt{1-x^2}} dx \ (I)$$

$$\Rightarrow I = -e^{\arcsin x}\sqrt{1-x^2} + xe^{\arcsin x} - I$$

$$\Rightarrow 2I = e^{\arcsin x}(x - \sqrt{1-x^2})$$

$$\Rightarrow \int \frac{xe^{\arcsin x}}{\sqrt{1-x^2}} dx = \frac{e^{\arcsin x}(x - \sqrt{1-x^2})}{2} + C$$

3) $$I = \int \frac{\ln^3 x}{x^2} dx$$

$$f(x) = \ln^3 x \quad g'(x) = \frac{1}{x^2}$$

$$f'(x) = \frac{3\ln^2 x}{x} \quad g(x) = -\frac{1}{x}$$

$$\Rightarrow I = -\frac{\ln^3 x}{x} + 3\int \frac{\ln^2 x}{x^2} dx$$

$$I_1 = \int \frac{\ln^2 x}{x^2} dx$$

$$f(x) = \ln^2 x \quad g'(x) = \frac{1}{x^2}$$

$$f'(x) = \frac{2\ln x}{x} \quad g(x) = -\frac{1}{x}$$

$$\Rightarrow I_1 = -\frac{\ln^2 x}{x} + 2\int \frac{\ln x}{x^2} dx$$

$$I_2 = \int \frac{\ln x}{x^2}dx$$

$$f(x) = \ln x \quad g'(x) = \frac{1}{x^2}$$

$$f'(x) = \frac{1}{x} \quad g(x) = -\frac{1}{x}$$

$$\Rightarrow I_2 = -\frac{\ln x}{x} + \int \frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x}$$

$$I = -\frac{\ln^3 x}{x} + 3\left[-\frac{\ln^2 x}{x} - 2\left(\frac{\ln x}{x} + \frac{1}{x}\right)\right] + C$$

$$\Rightarrow \int \frac{\ln^3 x}{x^2}dx = \frac{-\ln^3 x - 3\ln^2 x - 6\ln x - 6}{x} + C$$

4) $$I = \int \ln(x + \sqrt{1+x^2})dx \qquad \left(I = \int \int \frac{1}{\sqrt{1+x^2}}dx\right)$$

$$f(x) = \ln(x + \sqrt{1+x^2}) \quad g'(x) = 1$$

$$f(x) = \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \quad g(x) = x$$

$$\Rightarrow I = x\ln(x + \sqrt{1+x^2}) - \int \frac{\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}xdx \Leftrightarrow$$

$$\Leftrightarrow I = x\ln(x + \sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}}dx \Leftrightarrow$$

$$\Leftrightarrow I = x\ln(x + \sqrt{1+x^2}) - \int \left(\sqrt{1+x^2}\right)' dx \Leftrightarrow$$

$$\Leftrightarrow \int \ln(x + \sqrt{1+x^2})dx = x\ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C$$

$$5)\, I = \int e^x \sin x dx$$

$$f(x) = e^x \quad g'(x) = \sin x$$

$$f'(x) = e^x \quad g(x) = \cos x$$

$$\Rightarrow I = e^x \cos x - \int e^x \cos x dx$$

$$I_1 = \int e^x \cos x dx$$

$$f(x) = e^x \quad g'(x) = \cos x$$

$$f'(x) = e^x \quad g(x) = -\sin x$$

$$\Rightarrow I_1 = -e^x \sin x + \int e^x \sin x dx$$

$$\Leftrightarrow I_1 = -e^x \sin x + I$$

$$\Rightarrow I = e^x \cos x + e^x \sin x - I$$

$$\Leftrightarrow I = \frac{e^x(\sin x + \cos x)}{2} + C$$

$$6)\, I = \int \sin(\ln x) dx$$

$$f(x) = \sin(\ln x) \quad g'(x) = 1$$

$$f'(x) = \frac{\cos(\ln x)}{x} \quad g(x) = x$$

$$\Rightarrow I = x\sin(\ln x) - \int \cos(\ln x) dx$$

$$I_1 = \int \cos(\ln x) dx$$

$$f(x) = \cos(\ln x) \quad g'(x) = 1$$

$$f'(x) = -\frac{\sin(\ln x)}{x} \quad g(x) = x$$

$$\Rightarrow I_1 = x\cos(\ln x) + \int \sin(\ln x)dx$$

$$\Rightarrow I_1 = x\cos(\ln x) + I$$

$$\Rightarrow I = x\sin(\ln x) - x\cos(\ln x) - I \Leftrightarrow$$

$$\Leftrightarrow I = \frac{x\sin(\ln x) - x\cos(\ln x)}{2}$$

$$\Leftrightarrow \int \sin(\ln x)dx = \frac{x[\sin(\ln x) - \cos(\ln x)]}{2} + C$$

$$7) I = \int x^2 \text{arctg}(3x)dx$$

$$f(x) = \text{arctg}(3x) \quad g'(x) = x^2$$

$$f'(x) = \frac{3}{1+9x^2} \qquad g(x) = \frac{x^3}{3}$$

$$\Rightarrow I = \frac{x^3}{3}\text{arctg}(3x) - \int \frac{x^3}{1+9x^2}dx$$

$$I_1 = \int \frac{x^3}{1+9x^2}dx$$

$$\frac{x^3}{1+9x^2} = \frac{x}{9} - \frac{1}{9}\cdot\frac{x}{9x^2+1}$$

$$\int \frac{x^3}{1+9x^2}dx = \int \frac{x}{9}dx - \frac{1}{9}\int \frac{x}{9x^2+1}dx \Leftrightarrow$$

$$\Leftrightarrow \int \frac{x^3}{1+9x^2}dx = \frac{x^2}{18} - \frac{1}{9}\int \frac{1}{18}\cdot\frac{18x}{9x^2+1}dx \Leftrightarrow$$

$$\Leftrightarrow \int \frac{x^3}{1+9x^2}dx = \frac{x^2}{18} - \frac{1}{9}\int \frac{1}{18}\cdot\frac{(9x^2+1)'}{9x^2+1}dx \Leftrightarrow$$

$$\Leftrightarrow \int \frac{x^3}{1+9x^2}dx = \frac{x^2}{18} - \frac{1}{162}\ln(9x^2+1) + C$$

$$I = \frac{x^3}{3}\text{arctg}(3x) - \frac{x^2}{18} + \frac{1}{162}\ln(9x^2+1) + C$$

Utilizarea metodei integrarii prin parti la calculul integralelor recurente

8) $I_n = \int \frac{1}{(a^2+x^2)^n}dx$

$$I_n = \int \frac{1}{(a^2+x^2)^n}dx = \frac{1}{a^2}\int \frac{a^2}{(a^2+x^2)^n}dx = \frac{1}{a^2}\int \frac{a^2+x^2-x^2}{(a^2+x^2)^n}dx$$

$$I_n = \frac{1}{a^2}\left[\int \frac{a^2+x^2}{(a^2+x^2)^n}dx - \int \frac{x^2}{(a^2+x^2)^n}dx\right] \Leftrightarrow$$

$$\Leftrightarrow I_n = \frac{1}{a^2}\left[\int \frac{1}{(a^2+x^2)^{n-1}}dx - \int \frac{x^2}{(a^2+x^2)^n}dx\right]$$

$$\int \frac{1}{(a^2+x^2)^{n-1}}dx = I_{n\text{-}1}$$

$$J = \int \frac{x^2}{(a^2+x^2)^n}dx = \int x \cdot \frac{x}{(a^2+x^2)^n}dx$$

$$\left(\frac{1}{(a^2+x^2)^{n-1}}\right)' = (1-n)(a^2+x^2)^{-n}2x = \frac{2x(1-n)}{(a^2+x^2)^n}$$

$$J = \frac{2}{1-n}\int x \cdot \frac{2x(1-n)}{(a^2+x^2)^n}dx$$

$$f(x) = x \qquad g'(x) = \frac{2x(1-n)}{(a^2+x^2)^n}$$

$$f'(x) = 1 \qquad g(x) = \frac{1}{(a^2+x^2)^{n-1}}$$

$$\Rightarrow J = \frac{2}{1-n}\frac{x}{(a^2+x^2)^{n-1}} - \int \frac{1}{(a^2+x^2)^{n-1}}dx \Leftrightarrow$$

$$\Leftrightarrow J = \frac{2}{1-n}\frac{x}{(a^2+x^2)^{n-1}} - I_{n-1}$$

$$\int \frac{x^2}{(a^2+x^2)^n}dx = \frac{2}{1-n}\frac{x}{(a^2+x^2)^{n-1}} - I_{n-1}$$

$$I_n = \frac{1}{a^2}\left[\int \frac{1}{(a^2+x^2)^{n-1}}dx - \int \frac{x^2}{(a^2+x^2)^n}dx\right] \Rightarrow$$

$$\Rightarrow I_n = \frac{1}{a^2}\left[I_{n-1} - \frac{2}{1-n}\frac{x}{(a^2+x^2)^{n-1}} + I_{n-1}\right] \Rightarrow$$

$$\Rightarrow I_n = \frac{1}{a^2}\left[2\cdot I_{n-1} - \frac{2}{1-n}\cdot\frac{x}{(a^2+x^2)^{n-1}}\right] + C$$

$$9)\ I_n = \int \frac{x^n}{\sqrt{a^2+x^2}}dx$$

$$I_n = \int x^{n-1}\cdot\frac{x}{\sqrt{a^2+x^2}}dx$$

$$f(x) = x^{n-1} \quad g'(x) = \frac{x}{\sqrt{a^2+x^2}}$$

$$f'(x) = (n-1)x^{n-2} \quad g(x) = \sqrt{a^2+x^2}$$

$$I_n = x^{n-1}\sqrt{a^2+x^2} - \int (n-1)x^{n-2}\sqrt{a^2+x^2}dx$$

$$I_n = x^{n-1}\sqrt{a^2+x^2} - (n-1)\int x^{n-2}\sqrt{a^2+x^2}dx$$

$$J = \int x^{n-2}\sqrt{a^2+x^2}dx$$

$$J = \int x^{n-2}\frac{a^2+x^2}{\sqrt{a^2+x^2}}dx = a^2\int \frac{x^{n-2}}{\sqrt{a^2+x^2}}dx + \int \frac{x^n}{\sqrt{a^2+x^2}}dx$$

$$J = a^2\cdot I_{n-2} + I_n$$

$$I_n = x^{n-1}\sqrt{a^2+x^2} - a^2\cdot I_{n-2} - I_n$$

$$I_n = \frac{x^{n-1}\sqrt{a^2+x^2} - a^2}{2}\cdot I_{n-2} + C$$

$$I_{nI_{n-2}} = \int \sin^n x \, dx$$

$$I_n = \int \sin^{n-1} x \sin x \, dx$$

$$f(x) = \sin^{n-1} x \quad g'(x) = \sin x$$

$$f'(x) = (n-1)\sin^{n-2} x \cos x \quad g(x) = -\cos x$$

$$I_n = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx$$

$$J = \int \sin^{n-2} x \cos^2 x \, dx = \int \sin^{n-2} x(1-\sin^2 x) \, dx$$

$$J = \int \sin^{n-2} x - \sin^n x \, dx = \int \sin^{n-2} x dx - \int \sin^n x dx$$

$$J = I_{n-2} - I_n$$

$$\Rightarrow I_n = -\cos x \sin^{n-1} x + (n-1)(I_{n-2} - I_n) \Leftrightarrow$$

$$\Leftrightarrow I_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n$$

$$\Leftrightarrow n \cdot I_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2}$$

$$\Leftrightarrow I_n = \frac{-\cos x \sin^{n-1} x + (n-1)I_{n-2}}{n} + C$$

10) $I_n = \int \frac{1}{\cos^n x} dx$

$$f(x) = \frac{1}{\cos^{n-2} x} \quad g'(x) = \frac{1}{\cos^2 x}$$

$$f'(x) = \frac{(2-n)\sin x}{\cos^{n-1} x} \quad g(x) = tgx$$

$$I_n = \frac{tg(x)}{\cos^{n-2} x} + (n-2)\int \frac{\sin x tgx}{\cos^{n-1} x}$$

$$J = \int \frac{\sin x \operatorname{tg} x}{\cos^{n-1} x} dx = \int \frac{\sin x}{\cos^{n-1} x} \cdot \frac{\sin x}{\cos x} dx \Leftrightarrow$$

$$\Leftrightarrow J = \int \frac{\sin^2 x}{\cos^n x} dx = \int \frac{1 - \cos^2 x}{\cos^n x} dx \Leftrightarrow$$

$$\Leftrightarrow J = \int \frac{1}{\cos^n x} dx - \int \frac{\cos^2 x}{\cos^n x} dx = \int \frac{1}{\cos^n x} dx - \int \frac{1}{\cos^{n-2} x} dx \Leftrightarrow$$

$$\Leftrightarrow J = I_n - I_{n-2}$$

$$\Rightarrow I_n = \frac{\operatorname{tg} x}{\cos^{n-2} x} - (n-2) I_n + (n-2) I_{n-2} \Leftrightarrow$$

$$\Leftrightarrow (n-1) I_n = \frac{\sin x}{\cos^{n-1} x} + (n-2) I_{n-2} \Leftrightarrow$$

$$\Leftrightarrow I_n = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} I_{n-2} + C$$